Algebra Qualifying Exam
Fall 2016

This exam is 8 hours long. Collaboration and aids are not allowed. Do as many problems as you can. Single complete solutions are better than several partial solutions. Do not reprove major theorems unless asked to do so, but when you use such theorems say so. In writing down partial solutions try to indicate the gaps as clearly as possible, so that one can see what you do and do not know.

(1) Up to isomorphism, how many abelian groups are there of order $2016 (= 2^5 \cdot 3^2 \cdot 7)$?

(2) Let R be a commutative ring, and suppose that I and J are ideals of R with $I + J = R$.

  (a) Prove the following form of the Chinese Remainder Theorem:
$$\frac{R}{I \cap J} \cong \frac{R}{I} \times \frac{R}{J}.$$

  (b) Prove that $IJ = I \cap J$.

(3) Suppose that K/F is a Galois extension of fields. For all $\alpha \in K$, the norm from K to F of α is
$$N_{K/F}(\alpha) = \prod_{\sigma \in \mathrm{Gal}(K/F)} \sigma(\alpha)$$
is a homomorphism, $N_{K/F} : K^\times \to F^\times$. (You may assume this.)

Let $p \in \mathbb{Z}$ be prime, and let $n \geq 1$.

  (a) For all $x \in \mathbb{F}_{p^n}$, prove that $N_{\mathbb{F}_{p^n}/\mathbb{F}_p}(x) = x^{\frac{p^n-1}{p-1}}$.

  (b) Assume the conclusion of part (a). Show that $N_{\mathbb{F}_{p^n}/\mathbb{F}_p} : \mathbb{F}_{p^n} \to \mathbb{F}_p$ is surjective.

(4) Let $C^\bullet = C$ be a bounded complex of finite rank free $\mathbb{Z}$-modules. Let A be an abelian group. Prove that there exists an injective map
$$H_n(C) \otimes_{\mathbb{Z}} A \to H_n(C \otimes_{\mathbb{Z}} A).$$
Is this always an isomorphism? If so, prove it; if not, exhibit a counterexample.

(5) Let k be a field, and let $\mathfrak{m}$ be a maximal ideal of $k[x]$. We call $\mathfrak{m}$ an L-**point** if and only if $k[x]/\mathfrak{m} \cong L$. (The terminology is taken from Algebraic Geometry). Determine the number of $\mathbb{F}_8$-points for $k = \mathbb{F}_2$.

(6) Let $p \in \mathbb{Z}$ be an odd prime, and let ζ be a primitive pth root of unity.

  (a) Show that $\mathbb{Q}(\zeta)$ has exactly one subfield, K, with $[K : \mathbb{Q}] = 2$.

  (b) Show that K is real ($K \subseteq \mathbb{R}$) if and only if $p \equiv 1 \pmod 4$.

(7) Let $p \in \mathbb{Z}$ be prime, let G be an elementary p-group, let X be a set of size n with $p \nmid n$, and suppose that G acts on X. Prove that the action has a fixed point.

(8) Let $A \in \mathrm{M}_n(\mathbb{C})$ be an $n \times n$ matrix. Let $R \subseteq \mathrm{M}_n(\mathbb{C})$ be the subalgebra generated by A. Show that R is commutative and that $\dim_{\mathbb{C}} R \leq n$.

(9) Let R be a commutative ring, and suppose that

$$0 \to F' \to F \to F'' \to 0$$

is a short exact sequence of finite rank free R-modules. Let d', d, d'' denote their respective ranks. Prove that there is an (non-canonical) isomorphism between the alternating powers

$$\textstyle\bigwedge_R^d F \cong \bigwedge_R^{d'} F' \otimes_R \bigwedge_R^{d''} F''.$$

(10) Classify all groups of order 21 up to isomorphism.